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Integral analysis applied to radial film flows

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Abstract

Radial thin film flows are obtained by the impingement of circular free liquid jets on surfaces. The surfaces may be in the form of a circular plate, cone or that of a sphere. These flows are governed by the effects of inertia, viscosity, gravity and surface tension. Based on the film Reynolds number and Froude number, a circular hydraulic jump can be obtained in such flows. In this paper a new integral method is proposed for such axisymmetric laminar flows. The boundary layer approximation is used. The equations are solved using a cubic velocity profile, considering the radial hydrostatic pressure gradient in the film flow. In the new approach the coefficients of the cubic profile depend on the pressure gradient and body force terms and are allowed to vary with radial distance. Thus for example, separation can be predicted. The effect of the jet Reynolds number, Froude number and the surface dimension is considered. For flows with the circular hydraulic jump, the region upstream and downstream of the jump is solved separately using the boundary condition at the surface edge.

Nomenclature

a, b, c coefficients in the velocity profile
a_0 radius of nozzle
$Fr_j = U_0^2/ga_0$ jet Froude number
g_x, g_y components of g in x and y direction
h_{cr} critical value of film thickness
$h(x)$ liquid film thickness at x
$I_1 = \int_0^1 f \, d\eta$
$I_2 = \int_0^1 f^2 \, d\eta$ constants obtained from velocity profile
$I_3 = \int_0^1 f^3 \, d\eta$
$I'_2 = \int_0^1 f'^2 \, d\eta$
$Q = 2\pi q$ flow rate
$Re_j = U_0 a_0/\nu$ jet Reynolds number
R_j hydraulic jump radius
R_p plate radius
R_s radius of sphere
$u/U(x) = f(\eta)$ assumed velocity profile
U_{cr} critical value of average film velocity
U_0 average jet velocity
$U(x)$ free surface velocity
x_0 radius of jet at point of impingement
x horizontal distance along the surface
y vertical distance perpendicular to the surface
$\eta = y/h(x)$ nondimensional vertical coordinate.

1. Introduction

The flow of liquids in thin films is frequently observed in every day life as in flow of rain water on window panes and roofs, flows in a kitchen sink etc. Thin liquid films also find applications in industry as in during evaporation or condensation on a solid surface in a compact heat exchanger or cooling tower, spin coating in metal industries, and impingement cooling of solid wall with a liquid jet.

Radial thin film flows form a special class of the film flows and are obtained when a circular free liquid jet impinges on a surface. The surface may be in form of flat circular disk, surface of a cone or that of a sphere. When such a jet impinges normally, the jet spreads out into a thin film flowing radially away from the stagnation point. The effects of inertia, viscosity, gravity and surface tension govern such flows. As the film spreads out, the film thickness decreases and then increases and the film flows under the action of an adverse hydrostatic pressure gradient. For flows over surfaces of cones and spheres, the flow accelerates under the action of gravity which balances

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the effects of the adverse pressure gradient. The adverse pressure gradient can cause separation of the film flow at some radial distance leading to formation of a circular hydraulic jump. The flow in the layer upstream of the jump may remain laminar for moderately large Reynolds numbers, but its velocity is seldom uniform, except possibly near the beginning of the layer, owing to the action of viscosity. The Froude number near the impact region of the jet is quite large, being of the order of the ratio of the head of the falling liquid to the size of the cross section of the jet.

One of the earliest works done on the study of such a radially spreading thin film flow on a horizontal surface was by Watson [1]. Using the boundary layer approximation, Watson developed a similarity solution in the region upstream of the jump. The jump location was obtained by knowing the downstream film height (set by a barrier) and using the jump condition. He solved for the laminar and (in approximate form) turbulent flow, neglecting the effect of the pressure gradient due to gravity. Earlier, Tani [2] had solved the boundary layer equation, with the gravitational pressure gradient term included using a parabolic velocity profile. He postulated that flow separation is due to an adverse hydrostatic pressure gradient which caused the hydraulic jump.

Several authors have solved for the flow both upstream and downstream of the jump with the gravity term included and by assuming some form for the velocity profile. The two solutions are matched at the jump by using the jump conditions. Thomas *et al.* [3] and Rehman *et al.* [4] assume a parabolic velocity profile for solving the flow in the region upstream and downstream of the hydraulic jump. They consider the effect of rotation and the pressure gradient term on the flow of radial and planar thin film flows. Bohr *et al.* [5] gave a scaling relation for the jump position for flow on infinitely large plates without a barrier. The boundary condition at the edge of the plate is of interest; critical flow (unity Froude number) or infinite slope of the film has been used as edge condition. Buyevich and Ustinov [6] assume a cubic velocity profile with constant coefficients. Analytical expressions for the flow upstream of the jump are obtained by neglecting the gravity term.

Radial thin film flows over cones and spheres obtained by circular jet impingement have received less attention. They may not be accompanied by a hydraulic jump. In addition to the effects of inertia and viscosity, the effect of gravitational acceleration is felt due to the inclination as well as changing height. Zollars *et al.* [7] developed an asymptotic solution for flow of a low Reynolds number thin film down a right circular cone. The solution incorporates the effects of surface tension.

Cerro *et al.* [8] studied the rapid thin film flows at relatively large Reynolds numbers and capillary numbers through solutions of the governing equations using Karman–Pohlhausen method. Solutions for dip coating, falling films and liquid wall jets are obtained. Ghim *et al.* [9] reported on the boundary conditions in rectangular and cylindrical co-ordinates for the free surface of a thin film flow. Higgins *et al.* [10] derived the exact differential and the integrodifferential equations for the shape and change of shape of one-dimensional free surface flows. These equations similar to integral equations for boundary layer type of flow can be solved by imposing the appropriate boundary conditions . Mudawar *et al.* [11] studied the mass and momentum transport in smooth falling laminar liquid films at relatively high Reynolds number. Velocity measurements using LDV and modeling via the integral approach were used to obtain the velocity of the thin falling liquid films.

The presence of a reversed flow region downstream of the hydraulic jump is further confirmed by experiments of Nakaryakov *et al.* [12] and Craik *et al.* [13]. Though the presence of such an adverse pressure gradient had been taken into account in the governing equations by many investigators, they solved the equations in integral form by assuming a parabolic velocity profile, where the coefficients in the profile are assumed constants being obtained from the boundary conditions or by a cubic velocity profile without the pressure gradient term. The effect of the adverse pressure gradient on the change in the velocity profile, and finally leading to separation has not been considered.

In this paper, integral relations of the momentum and mechanical energy for the flow of a thin radial liquid film formed by impingement of a circular liquid jet over a general axisymmetric body have been derived. These free surface film flows are nearly unidirectional, bounded, axisymmetric, viscous, and incompressible liquid flows at high Reynolds number, in which the pressure is substantially hydrostatic in the transverse direction. Viscous shear is virtually absent at the free surface and the capillary forces are negligible. The velocity, pressure and film thickness are governed by viscous, inertial and gravity (or other external) forces only. In all the previous works on the integral analysis of the film flows the coefficients in the assumed polynomials for the velocity profile have been assumed to be constants. The new feature in the present paper is that the coefficient values depend on the pressure gradient and the body force terms and are allowed to vary with radial distance. Also, these integral equations differ from those normally used in boundary layer flows in the sense that the coefficients in the assumed cubic velocity profile are unknown *a priori* and have to be determined along the flow direction. The pressure is not impressed on the flow from the external inviscid flow as in the case of normal boundary layer flows, but is purely hydrostatic to be determined from the varying film thickness. The effect of varying jet Reynolds number, Froude number and the surface radius on the film thickness and surface velocity of the thin film is considered. The rest of the paper is outlined as follows. In Section 2 the flow being modeled is considered and the integral equations for both the developing and the fully developed region are derived. Also the condition at the edge of the surface is derived. In Section 3 some representative results for flow over a flat plate, cone and sphere are presented.

2. Problem formulation

In this section, a detailed description of the problem is given and the integral relations for the laminar high Reynolds number radial flow of a thin liquid film over axisymmetric surfaces are derived. Flows at very low Reynolds number such as creeping flows are not considered here. The laminar film flows are obtained by the impingement of circular laminar free liquid jets on surfaces.

As shown in Fig. 1 the flow regime of the jet impinging on a general axisymmetric body can be divided into the following parts.

- Region 1: The stagnation point region whose dimensions are of the order of the jet radius, x_0. Within this region the main flow velocity grows rapidly from zero to the undisturbed flow velocity U_0.
- Region 2: The region of boundary layer type of flow for $x > x_0$, where the boundary layer is developing and the velocity outside the boundary layer is dependent on the flow geometry.
- Region 3: The region of fully developed flow, where the viscous effects are felt up to the free surface.
- Region 4: The region just before the hydraulic jump where the gravity effects are important and an adverse pressure gradient is present.
- Region 5: The region of the hydraulic jump including the separation eddy.
- Region 6: The region of the flow downstream of the hydraulic jump. Further the flow is assumed to remain laminar in the region both upstream and downstream of the separation region.

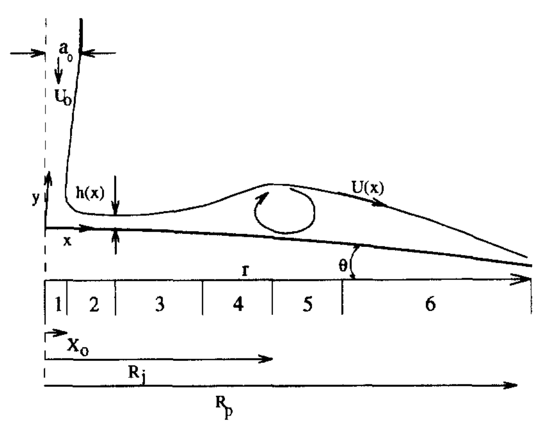

Fig. 1. Schematic of radial thin film flow over an axisymmetric surface.

We note that viscous effects are felt up to the free surface downstream of region 2. Upstream of the jump the flow is supercritical and subcritical at the downstream. In regions 4, 5 and 6, the local Froude number is of order one and gravity is important. Not all regions may exist in a given flow case. If the surface radius is small enough a hydraulic jump will not exist. For flows over surfaces of cone and sphere the hydraulic jump might not exist and the flow may remain supercritical until the edge of the surface.

3. Viscous analysis

The viscous analysis is carried out for an incompressible, steady, axisymmetric and laminar flow of a thin liquid film. Invoking the boundary layer approximation the equations are:

Continuity:

$$\frac{\partial(ur)}{\partial x}+\frac{\partial(vr)}{\partial y}=0 \tag{1}$$

Momentum:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=v\frac{\partial^2 u}{\partial y^2}-g_y\frac{\mathrm{d}h}{\mathrm{d}x}+g_x. \tag{2}$$

For the case of flow over the surface of a cone:

$$r=x\cos\theta;\quad g_x=g\sin\theta;\quad g_y=g\cos\theta. \tag{3}$$

For a sphere we replace r by,

$$r=R_s\sin\left(\frac{x}{R_s}\right) \tag{4}$$

and g_x and g_y by,

$$g_x=g\sin\left(\frac{x}{R_s}\right);\quad g_y=g\cos\left(\frac{x}{R_s}\right). \tag{5}$$

The pressure term in the standard boundary layer equation is replaced by the hydrostatic pressure variation along the radial direction. This term is obtained from the y momentum equation on the assumption that the liquid film is shallow and that the vertical velocities are small compared to streamwise velocities and that the vertical variations (through the layer) are much more rapid than those in the streamwise direction.

4. Developing region

We first consider the flow in the region $x > x_0$. Here the streamwise pressure gradient is assumed to be practically absent and velocity outside the boundary layer is assumed to be constant being equal to U_0. Further we assume that the length of the region is small enough so that the effects of gravitational acceleration can be neglected.

For case of flow over a sphere we assume that the developing region of the flow, which in this case will be region very close to the stagnation point, can be approximated to be of the order of the jet radius. Thus the equations for the fully developed flow region are valid from just near the stagnation point region.

The final form of the momentum integral equations are arrived at by integrating the boundary layer equations without the pressure gradient and the gravitational acceleration terms, from 0 to $\delta(x)$ the boundary layer thickness, and noting that velocity is continuous at $\delta(x)$. The resulting equations are:

Continuity:

$$Q = 2\pi r\left[\int_0^{\delta(x)} u\,\mathrm{d}y + U_0(h(x)-\delta(x))\right] \tag{6}$$

Momentum:

$$\frac{1}{r}\int_0^{\delta(x)} \frac{\mathrm{d}}{\mathrm{d}x}(u^2 r)\,\mathrm{d}y - \frac{U_0}{r}\int_0^{\delta(x)} \frac{\mathrm{d}}{\mathrm{d}x}(ur)\,\mathrm{d}y = -\nu\left.\frac{\partial u}{\partial y}\right|_{y=0} \tag{7}$$

Boundary Conditions:

$u(x,0) = 0$

$u(x,y) = U_0 \quad \text{at } y = \delta(x)$

$\frac{\partial u}{\partial y} = 0 \quad \text{at } y = \delta(x).$

We use an additional condition, as in the Karman–Pohlhausen method of

$$\nu\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} = 0.$$

This is derived by applying the momentum equation (2) at the wall.

Integral analysis requires the choice of the appropriate velocity profiles. The above integral equations are solved by assuming a cubic velocity profile of the form,

$$\frac{u}{U_0} = f(\eta) = a\eta + b\eta^2 + c\eta^3 \tag{8}$$

the coefficients a, b, c are obtained from the boundary conditions and are equal to 1.5, 0.0, -0.5 respectively. Substituting the above velocity profile into the integral equations and calculating the momentum flux and the viscous shear terms and using the continuity equation, we obtain a relationship for the development of the boundary layer and height of the liquid film with radius which are of the following form for the horizontal plate,

$$\delta(x) = 4.736 x_0\left[\frac{x\nu}{Q}\right]^{1/2} \tag{9}$$

$$h(x) = \frac{x_0^2}{2x} + 0.375\delta(x). \tag{10}$$

The above equations are equivalent to that derived by Watson [1] and give an explicit relation for the film thickness and the boundary layer thickness. These relations are valid till the boundary layer reaches the free surface where $\delta(x) = h(x)$ and this occurs at a radius given by,

$$x = 0.3054\left[\frac{x_0^2 Q}{\nu}\right]^{1/3}. \tag{11}$$

The above set of equations for the developing region of the flow are assumed to be valid for all configurations. This assumption does not strictly apply to the flow over a cone or over a sphere where there is acceleration of the free stream.

5. Fully developed region

Next we consider the flow in the region where the boundary layer thickness is of the order of the film thickness and hence the viscous effects are felt up to the free surface (region 3). For the fully developed region the integral equations are derived from the boundary layer equations including the pressure gradient term and the gravitational acceleration term. The final form of the momentum integral equations are arrived at by integrating both sides of the governing equations from 0 to $h(x)$ and noting that shear stress at the free surface is zero. This is true as long as the effects of surface tension due to curvature effects and shear stress due to air at the free surface are small.

Continuity:

$$Q = 2\pi r\int_0^{h(x)} u\,\mathrm{d}y \tag{12}$$

Momentum:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{h(x)} (u^2 r)\,\mathrm{d}y = -\nu\left.\frac{\partial u}{\partial y}\right|_{y=0} - g_y\frac{\mathrm{d}h}{\mathrm{d}x}h(x) + g_x h(x) \tag{13}$$

Boundary Conditions:

$$u|_{y=0} = 0 u(x,y) = U(x) \text{ at } y = h(x)\frac{\partial u}{\partial y} = 0 \quad \text{at } y = h(x) \tag{14}$$

Additional Condition:

$$\nu\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} = g_y\frac{\mathrm{d}h}{\mathrm{d}x} - g_x. \tag{15}$$

The mechanical energy integral equation are derived for the flow of a radial thin film on a general axisymmetric body. The energy integral analysis is instructive as it is possible to study each of the kinetic energy, potential energy and viscous dissipation terms separately. The energy integral analysis is restricted to the fully developed region. The energy integral equations are derived from the governing equations by the scalar multiplication of the momentum equation with the tangential velocity component and then integrating over the liquid film height from 0 to $h(x)$.

Energy:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{h(x)}\frac{u^3 r}{2}\mathrm{d}y = -\nu r\int_0^{h(x)}\left(\frac{\partial u}{\partial y}\right)^2\mathrm{d}y - g_y\frac{\mathrm{d}h}{\mathrm{d}x}q + g_x q. \quad (16)$$

As for the case of the developing boundary layer region we again assume a cubic velocity profile (8). The coefficients a, b, c are obtained from the boundary conditions. It must be noted that they are not constants, being proportional to the streamwise derivative of height and the gravitational acceleration term. The coefficients derived from the conditions take the form:

$$a = 1.5 - \frac{b}{2} \quad (17)$$

$$c = -0.5 - \frac{b}{2} \quad (18)$$

$$b = \frac{g_y h(x)^2}{2\nu U(x)}\frac{\mathrm{d}h}{\mathrm{d}x} - \frac{g_x h(x)^2}{2\nu U(x)}. \quad (19)$$

For the case of flow over a flat circular plate and for flow over surfaces of cones with small cone angles, separation can take place at some radial distance in the developed region of the flow regime due to the presence of an adverse pressure gradient. As the governing equations are no longer valid after the region of separation it is necessary to predict the film height and radius at separation. The criterion for steady separation,

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0 \quad (20)$$

i.e., zero wall shear stress, gives the value of the constants a, b, c in the velocity profile to be 0.0, 3.0, -2.0 respectively.

For large cone angles and for flow over the surface of sphere the pressure gradient term can be neglected as compared to the acceleration term. For such cases, using the continuity equation and the expression for b from the boundary condition we get a cubic equation for b.

Without the pressure gradient term and using the integral form of the continuity equation we get:

$$b = -\frac{g_x q^2}{2\nu U(x)^3 r^2 I_1^2} \quad (21)$$

where

$$I_1 = \frac{3.0 - b}{2}. \quad (22)$$

Substituting for I_1 in terms of b we get,

$$\frac{25b}{32} - \frac{5b^2}{48} + \frac{b^3}{288} = \frac{q^2 g_x}{\nu U(x)^3 r^2}. \quad (23)$$

Since for the flow downstream of the hydraulic jump the film thickness and the gradients in height are large it is not possible to solve the equations downstream of the jump using a velocity profile with variable coefficients. We choose a velocity profile for solutions of the equations which corresponds to the coefficient in the velocity profile b being equal to zero. With this simpliciation the coefficients a and c are equal to 1.5 and -0.5 respectively.

6. Integral equations

Substituting the assumed velocity profiles into the equations we obtain the integral equations for the fully developed region in the dimensional form as follows:

Continuity:

$$Q = 2\pi r h U(x) I_1 \quad (24)$$

Momentum:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{U(x)I_2}{I_1}\right] = -\frac{\nu}{I_1 h(x)^2}(f_0'' + f_0') + \frac{g_x}{U(x)I_1} \quad (25)$$

where

$$f_0'' = \frac{g_y h^2}{\nu U(x)}\frac{\mathrm{d}h}{\mathrm{d}x} \quad \text{and} \quad f_0' = \frac{U(x)}{h}\left.\frac{\partial u}{\partial y}\right|_{y=0}$$

and in terms of coefficients of the velocity profile: $f_0'' = 2b$; $f_0' = a = 1.5 - b/2$;

Energy:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{U(x)^2 I_3}{2I_1} + g_y h(x)\right] = -\nu\frac{I_2' U(x)}{I_1 h(x)^2} + g_x. \quad (26)$$

The first term in the energy integral equation corresponds to the kinetic energy, the second to the potential energy and the third term is the energy dissipation term. In all the results presented in this paper the continuity and momentum equations are solved. Of course, the energy equation can be used instead of the momentum equation. I_1, I_2, I_3, I_2' are values of the integral obtained from the assumed velocity profile. The values of the various integrals as a function of b, the coefficient in the velocity profile is given in the Table 1.

7. Surface edge boundary condition

Downstream of the jump region the flow reattaches as reported by Craik *et al.* [13]. Further downstream of the separation region the thin film falls freely at the edge of the plate. The boundary condition at the plate edge has an effect on the flow downstream of the separation region. This boundary condition in turn depends on the plate radius. If the plate radius is sufficiently small such that no separation occurs, then the flow regime remains supercritical and the edge boundary condition has no effect on the flow upstream. If the plate radius is sufficiently large

Table 1
Values of integrals of velocity profile for various values of b (coefficient in velocity profile)

b	I_1	I_2	I'_2	I_3	Velocity profile
−1.0	0.666	0.533	1.133	0.45715	Parabolic
0.0	0.625	0.485	1.2	0.409375	Zero pressure gradient
1.0	0.5833	0.44285	1.133	0.36875	Adverse pressure gradient
2.0	0.54166	0.404761	1.1333	0.335275	Adverse pressure gradient
3.0	0.5	0.37142	1.2	0.3071	Separating

to allow for separation to take place due to an adverse pressure gradient the flow downstream of the separation region flows under the dominant action of gravity and viscosity and remains subcritical, before accelerating as it falls freely from the edge of the plate. The edge boundary condition now influences the flow downstream of the separation region.

We solve the energy integral equation for the flow downstream of the separation region. The plate radius is assumed to be sufficiently large enough to allow for separation to occur at some radial location on the plate.

We propose that at a certain radius, corresponding to the plate radius, the flow downstream of the separation region reaches a state of minimum energy given by,

$$\frac{dE}{dh} = 0 \tag{27}$$

where h is the film thickness. That is, the flow adjusts itself such that maximum dissipation of the initial energy occurs. Here E corresponds to the (L.H.S.) term in brackets of (24). For minimum energy,

$$\frac{dE}{dh} = -\frac{q^2 I_3}{r^2 I_1^3 h^3} + g_y = 0. \tag{28}$$

For minimum energy at any $x = x_p$, x_p being the plate radius, the critical value of the film thickness is given by,

$$h_{cr} = \frac{q^{2/3} I_3^{1/3}}{x_p^{2/3} I_1 g_y^{1/3} \sin^{2/3}\theta}. \tag{29}$$

The Froude number based on average velocity, corresponding to this case of minimum energy is,

$$Fr_{cr} = \frac{U_{cr}^2}{g h_{cr}} = \frac{I_1^3}{I_3} \tag{30}$$

which for the assumed cubic velocity for the energy integral takes the value of 0.59 and for a uniform profile the value of 1.0.

For case of flow over the surface of a sphere, where the flow is always accelerated, separation might not occur over the surface. In such cases the flow remains supercritical all way till it flows down from the surface, and the edge boundary condition has no effect on the flow upstream.

8. Numerical results and discussion

In this section the solution of the integral equations (continuity, momentum and the equation for b) are presented for flow over a flat circular plate, the surface of a cone and that of a sphere. The integral equations are nondimensionalized using the radius of the nozzle for the length scale and the average jet velocity for the velocity scale. The resulting equations have the jet Reynolds number and Froude number as nondimensional parameters. The jet Reynolds number, Froude number and the plate radius used in the results correspond to some actual values in the experiments of Rao [14]. The equations are solved iteratively for b, till convergence is achieved. The fourth order Runge Kutta method is used for solving the equations. Flow downstream of the jump is solved using the shooting method with the critical film thickness at the plate edge as the boundary condition. The results here are presented in terms of nondimensional variables.

9. Flow over a horizontal plate

The flow of a thin radial liquid film on a horizontal circular plate is considered here. The jet Reynolds number Re_j, the Froude number Fr_j and the plate radius R_p play a dominant role in affecting the flow characteristics of the radial liquid film. Figure 2 shows the variation of the nondimensional film thickness versus the plate radius from the stagnation point of the circular jet impingement for a particular value of the jet Reynolds number, Froude number and the plate radius. As the jet impinges on the flat plate, a thin liquid film is obtained, which spreads out radially from the stagnation point region. As the boundary layer is still developing the viscous effects are not felt up to the free surface, the liquid film thickness decreases initially with radial distance, similar to the inviscid supercritical film. The boundary layer reaches the free surface at a certain radius and the viscous effects are felt up to the free surface. Further downstream the film thickness now starts increasing as the velocity decreases due to viscous retardation. At a certain radial

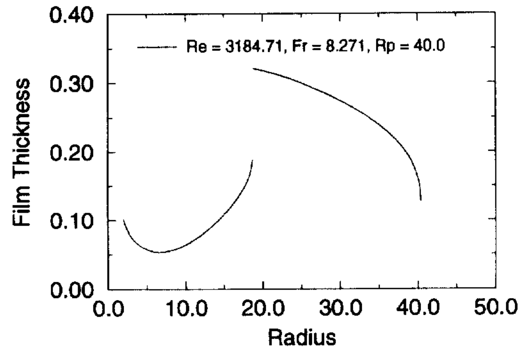

Fig. 2. Nondimensional film thickness versus radius at $Re_j = 3184.71$, $Fr_j = 8.271$ and $R_p = 40.0$.

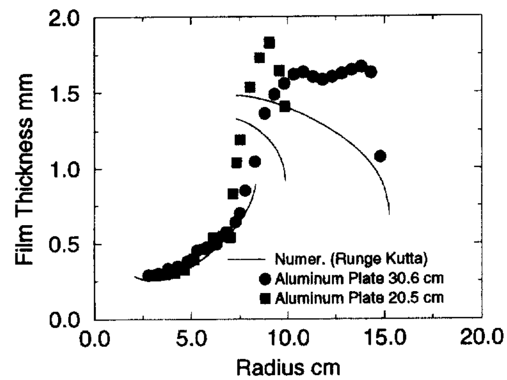

Fig. 3. Comparison of the measured and computed liquid film thickness upstream and downstream of the hydraulic jump. Flow rate = 2.5 lpm.

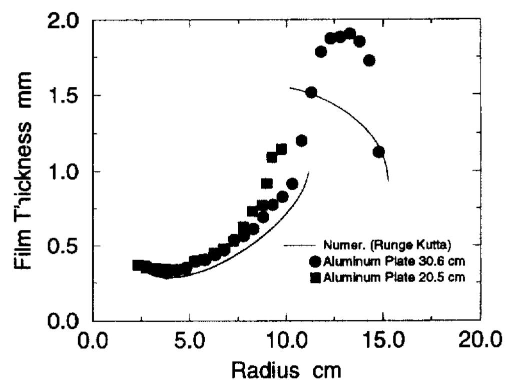

Fig. 4. Comparison of the measured and computed liquid film thickness upstream and downstream of the hydraulic jump. Flow rate = 4.0 lpm.

distance, the liquid film thickness increases very sharply leading to separation, corresponding to $b = 3.0$. The presence of the gravitational pressure gradient term in the governing momentum equation provides an adverse pressure gradient to the flow. The viscous retardation coupled with this adverse pressure gradient leads to separation at a certain radius.

The radius of separation represents a discontinuity in the liquid film thickness, connecting the supercritical region to the subcritical region. The flow can get over this discontinuity through a 'shock' or a hydraulic jump, corresponding to a critical Froude number. Thus it requires matching of the flow downstream of the hydraulic jump to the flow upstream. Just downstream of the jump a reversed flow region is present. The thin liquid film flows over this separation bubble and reattaches further downstream. Such a flow structure is not captured by the integral analysis. The hydraulic jump in experiments is more a gradual change in the height from the upstream supercritical region to the downstream subcritical region rather than a discontinuity in height. The flow downstream of the hydraulic jump is under the dominant action of gravity and viscosity. The plate edge boundary condition of critical flow [see equation (27)] at the plate edge determines the height of the film just downstream of the matched hydraulic jump region.

Figures 3 and 4 show the comparison of the liquid film thickness measured for flow rates of 2.5 lpm and 4.0 lpm respectively, for two different sizes of the aluminum plate from experiments of Rao [14]. The measured values are also compared with the film thickness variation obtained through integral analysis for the flow of such a radial thin liquid film. The figures show the effect of the plate radius on the height of the liquid film downstream of the hydraulic jump. For the smaller aluminum plate the hydraulic jump is pushed upstream and the effect of the plate edge results in a higher liquid film thickness downstream of the hydraulic jump. The measured liquid film thickness matches well with the liquid film variation obtained through integral analysis for the region upstream of the hydraulic jump. Downstream of the hydraulic jump there is a large difference in the measured and the computed values of the liquid film thickness and also in the behavior of the film height variation for the two different aluminum plates. The difference between the theory and the computations can be attributed to the following approximations involved in the integral analysis for the thin film flow.

- As the flow separates in the region of the hydraulic jump and reattaches further downstream, the governing equations are not strictly valid in the region of the hydraulic jump. The effects of surface tension become important in the region of the hydraulic jump due to the large effects of curvature. Such effects are not taken into account in the integral analysis. Under actual flow conditions the hydraulic jump involves a gradual change in the liquid film thickness from the supercritical region to the subcritical region. Integral analysis assumes a sharp discontinuity in the height at the separation radius.

- Capillary gravity waves are found downstream of the hydraulic jump and carry with them energy and momentum. This contribution is not accounted for in both the integral analysis and in the derivation of the jump conditions.
- We assume that the flow is choked at the plate edge. This assumption may not always be valid. In addition, surface tension effects at the plate edge, which may be significant, are also neglected.
- The analysis is not valid in the separated region of the flow.

The liquid film thickness upstream of the hydraulic jump is free of any of the effects mentioned above and this could be the reason for the approximate integral analysis matching well with the measured values.

In Fig. 5 the free surface velocity is plotted versus the radial distance and the result compared with that of Watson [1] for the region upstream of the hydraulic jump. The free surface velocity decreases gradually from the average jet velocity in the developing region as we move towards the hydraulic jump in the fully developed region due to viscous retardation. The variation of the free surface velocity versus the radial distance downstream of the hydraulic jump is shown in Fig. 6. As contrary to the flow upstream of the hydraulic jump, the free surface velocity decreases and then increases at the edge of the plate, indicating acceleration from a subcritical to a critical flow region at the plate edge.

In Fig. 7, b, the coefficient in the velocity profile is plotted versus the radial distance. The value of b, changes from a negative value indicating a decreasing height, to a value of three, indicating separation at a particular radius. The velocity profile upstream of the hydraulic jump, changes from a Blasius type of profile in the developing region to one with an inflection point in the adverse pressure gradient region of the flow. Such variations are made possible in this analysis as the coefficients in the velocity profile are not constant and vary along the flow direction.

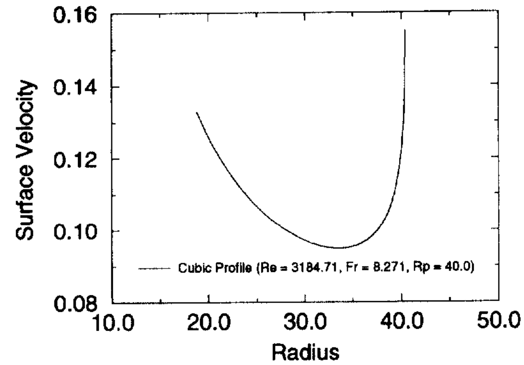

Fig. 6. Nondimensional surface velocity versus radius, downstream of the hydraulic jump at $Re_j = 3184.71$, $Fr_j = 8.271$, $R_p = 40.0$.

As mentioned earlier the jet Froude number, the Reynolds number and the plate radius affect the flow characteristics. Keeping the jet Reynolds number, plate radius fixed and varying the Froude number, the location of the separation point moves downstream with an increase in the jet Froude number. The film thickness shows only a little change with Froude number as we near the jump region. This is because the effect of gravity is not important in this region and the viscous forces are not changing for this case of constant Reynolds number of the jet. In contrast the flow downstream of the hydraulic jump is under the dominant action of gravity and viscosity. We see an appreciable change in the film thickness as the jet Froude number is changed. The film height increases with an increasing Froude number and decreasing effect of gravity.

Keeping the jet Froude number and the plate radius fixed, with a decrease in Reynolds number the effects of viscosity is dominant in the region upstream of the

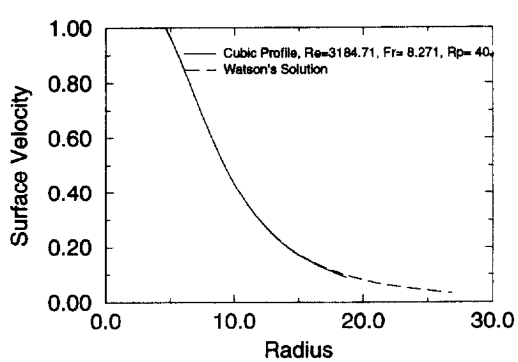

Fig. 5. Nondimensional surface velocity versus radius, upstream of the hydraulic jump at $Re_j = 3184.71$, $Fr_j = 8.271$, $R_p = 40.0$.

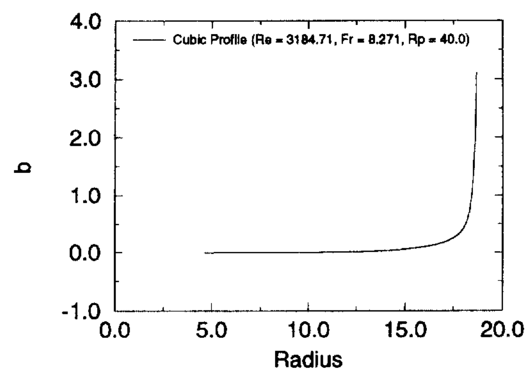

Fig. 7. b, Coefficient in velocity profile versus radius, upstream of the hydraulic jump at $Re_j = 3184.71$, $Fr_j = 8.271$, $R_p = 40.0$.

hydraulic jump leading to separation at a much smaller radius compared to higher values of Re_j. Downstream of the hydraulic jump a similar behavior continues and decreasing the Re_j increases the film thickness due to increased viscous retardation. If the Reynolds number of the jet is sufficiently large separation can totally be avoided over the given plate length and the thin film can be supercritical film till the plate edge.

When the jet Reynolds number and the Froude number are held constant but the plate radius is varied, for a sufficiently small plate radius no separation is obtained on the plate as in Fig. 4. Increasing the plate radius leads to separation at a certain radius over the plate. A larger plate radius corresponds to a higher film thickness just downstream of the hydraulic jump (see Fig. 3 also). This can be explained as an increase in the plate radius leads to a corresponding decrease in the critical value of the film thickness at the plate edge [see equation (27)]. A decrease in the critical value of the height leads to an increase in energy just downstream of the jump, in the subcritical region, corresponding to an increased height of the film thickness just downstream of the hydraulic jump.

10. Flow over surface of a cone

For the radial thin film flow over the surface of a cone the effects of gravitational acceleration and the gravitational adverse pressure gradient counteract each other depending on the cone angle.

Figure 8 shows the plot of the liquid film thickness versus the radial distance for a particular value of the jet Reynolds number, the Froude number, the length of the surface of the cone and different values of the cone angle. Radial film flow over the surface of the cone can occur without any flow separation. For the cone angles considered, the gravitational acceleration is much more dominant that the gravitational adverse pressure gradient. This accelerates the flow as the liquid film thickness increases due to viscous retardation there by avoiding separation in the flow. There is no discontinuity in height at any radius as for the flow over a flat plate case. The liquid film thickness is continuous from the stagnation point till the edge of the plate.

The free surface velocity decreases continuously from the stagnation point till the edge of the plate. Compared to the radial flow over a flat plate the decrease in the free surface velocity is much less due to the acceleration of the flow due to the dominant gravitational acceleration term in the governing equation. Figure 9 shows such a variation of the free surface velocity versus the radial distance.

The jet Reynolds number, Froude number, the cone radius and the cone angle are independent parameters which affect the plate radius. Increasing the cone angle accelerates the flow, and the separation which takes place for flow over a flat plate can be avoided and the thin film flows as a supercritical flow till the edge of the plate.

11. Flow over surface of a sphere

The flow of a thin liquid film over the surface of a sphere obtained by the impingement of a circular free laminar liquid jet forms a class of divergent convergent radial film flow. We neglect the gradient in film height term in the governing equations. This approximation is valid if the sphere radius is not large compared to the jump radius on a horizontal plate.

Figure 10 shows the variation of the film thickness versus angle in degrees measured from one pole to other for a particular value of the jet Reynolds number, the Froude number and the sphere radius. Due to a divergent sphere surface from an angle of 0–90° the area of the film flow increases and correspondingly the film height decreases. The free surface velocity as shown in Fig. 11

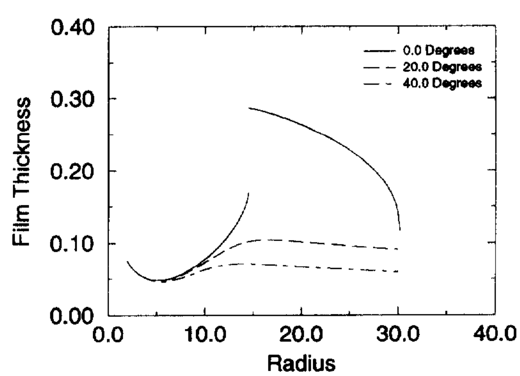

Fig. 8. Nondimensional film thickness versus radius at $Re_j = 3184.71$, $Fr_j = 8.271$ and $R_p = 30.0$.

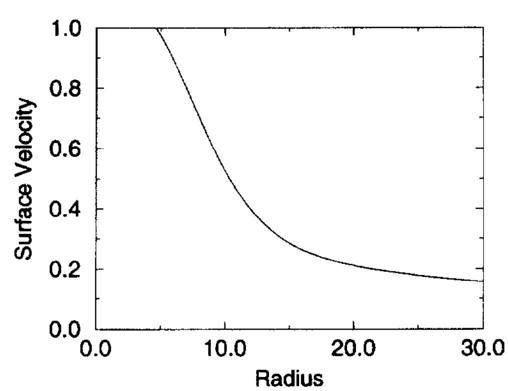

Fig. 9. Nondimensional surface velocity versus radius at $Re_j = 3184.71$, $Fr_j = 8.271$, $R_p = 30.0$, $\theta = 35.0°$.

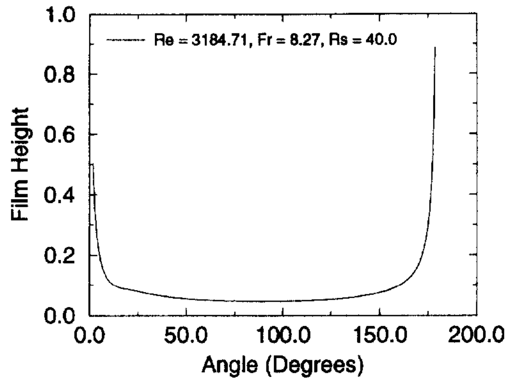

Fig. 10. Nondimensional film thickness versus angle measured from pole of the sphere at $Re_j = 3184.71$, $Fr_j = 8.271$ and $R_s = 40.0$.

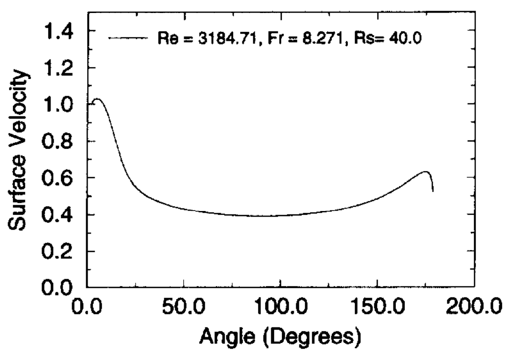

Fig. 11. Nondimensional surface velocity versus angle measured from pole of the sphere at $Re_j = 3184.71$, $Fr_j = 8.271$ and $R_s = 40.0$.

also decreases as the angle changes from 0–90°. From 90–180° the sphere has a converging surface and a decreasing area and to satisfy the continuity requirements the film thickness and the surface velocity increases. The present analysis breaks down near the angle of around 180° where the liquid film detaches from the sphere surface and falls down freely.

As compared to the flow over a flat plate, the flow over the surface of the sphere always remains supercritical. The Froude number increases from 0–90°. From 90–180° the Froude number becomes negative due to the opposite direction of the gravitational acceleration. The Reynolds number of the flow decreases sharply from an angle of 0° and reaches a minimum value around 90°. It further gradually increases from 90–180°, corresponding to a considerable increase in the film thickness and a small increase in the surface velocity. The value of b, the coefficient in the velocity remains constant over almost the whole sphere angle having a value of around -1.2. For case of a parabolic velocity profile the value of $b = -1.0$. The velocity profile thus for the whole region of the flow remains very nearly parabolic.

As for the case of the flat plate, Re_j, Fr_j and sphere radius are important independent parameters affecting the flow. Increasing the Fr_j i.e. decreasing the effects of gravity increases the film thickness over the whole surface of the sphere. Keeping the value of the Fr_j and the sphere radius constant but increasing the Re_j decreases the effects of viscosity and in turn makes the film thinner. For a fixed value of Re_j and Fr_j increasing the sphere radius increases the change in area encountered over the diverging and converging part of the sphere, there by decreasing the overall film thickness.

12. Conclusions

In this paper integral equations for the momentum and energy have been derived from the respective governing equations for the general axisymmetric flow of thin liquid films. A method of solution by assuming a cubic velocity profile for the film flow is proposed and applied to a few specific configurations. The coefficients of the polynomial, and thus the shape factor of the velocity profile, are allowed to vary with radial distance.

For a radial thin film flow on a circular flat plate, based on the jet Reynolds number and the Froude number, separation of the thin liquid film flow can occur at a certain radial distance. Generally a circular hydraulic jump precedes the separation point and connects the upstream supercritical flow regime to the downstream subcritical flow. The plate edge boundary condition of minimum energy of the thin liquid film, affects the flow downstream of the hydraulic jump. For the case of flow over the surface of a cone, the cone angle θ determines the dominance of the gravitational acceleration term over the adverse pressure gradient term. For large cone angles, a hydraulic jump is totally avoided and the thin film flows as a supercritical film all the way till the plate edge. Thin film flow over a sphere remains supercritical in all the cases considered all the way from the top pole of the sphere to the bottom pole. The flow occurs under the dominant action of the gravitational acceleration, and the velocity profile remains nearly parabolic for the whole region of the film flow.

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